# From directional statistics to applications in signal processing

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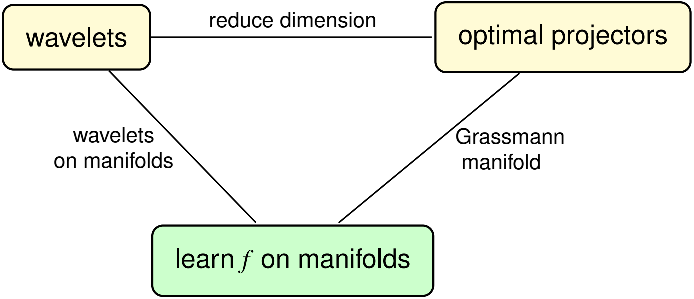
Faculty of Mathematics

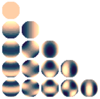
Advances in Directional Statistics
Brussels 2014

VIENNA SCIENCE
AND TECHNOLOGY FUND

# High-dimensional harmonic analysis

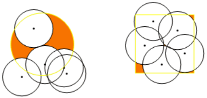



eigenfunctions of Laplace-Beltrami



Laplacian (eigenfunctions) on manifolds

covering numbers



# Vienna Research Group

Manuel Gräf

Monika Dörfler

2008: 3-year Fellowship at National Institutes of Health (NIH)
High-Resolution Mapping of Retinal Chromophores

2010: NIH/DFG Research Career Transition Awards Program

Developing Effective Algorithms for High-dimensional Biomedical Data and Image Analysis through Dimension Reduction Methods and Sparse Data Representations

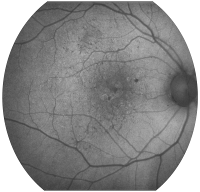
Deutsche Forschungsgemeinschaft DFG

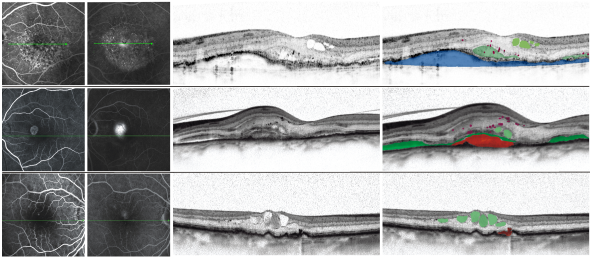
2013: Wiener Wissenschafts-, Forschungs- und Technologiefonds

Vienna Research Groups for Young Investigators Call 2012 “Mathematics and ..."
Computational Harmonic Analysis of High-Dimensional Biomedical data

WWTF VIENNA SCIENCE AND TECHNOLOGY FUND

# 8-year project with the Eye Clinic Vienna





# Outline

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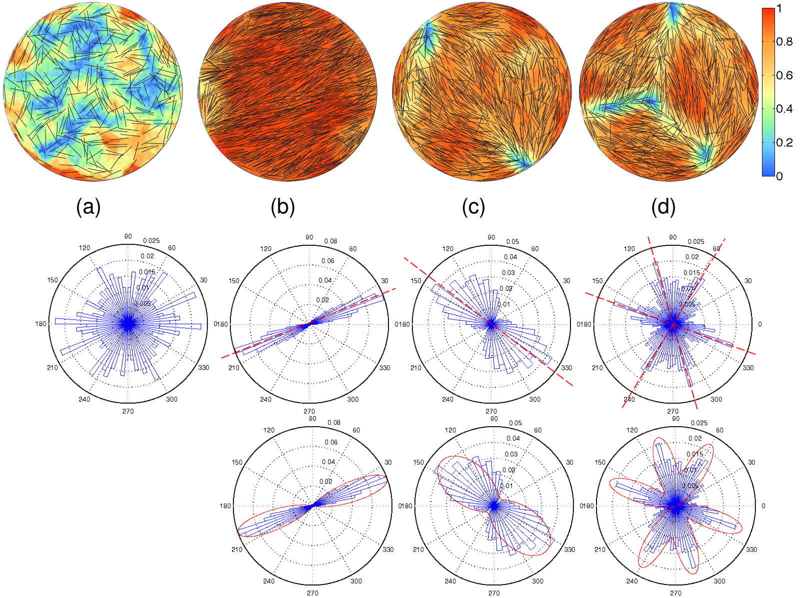
# Which one is more ordered?

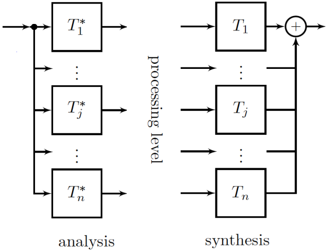
(Jennifer Galanis)


1
0.8
0.6
0.4
0.2
0
(a)
(b)
(c)
(d)

# Outline

# filter bank design

Given $\{T_j\}_{j=1}^{n} \subset \mathbb{C}^{d \times r}$:




**Requirements:**

- perfect reconstruction: $z = \sum_{j=1}^{n} T_j T_j^* z, \forall z \in \mathbb{R}^d$
- equal norm: $\|T_j\|_{HS} = \|T_k\|_{HS} \quad \forall j, k$

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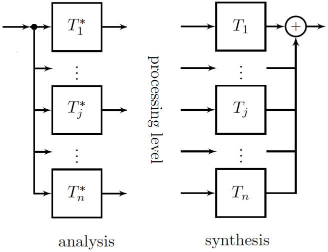


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# 2 naive ideas

For "nice" $\{T_j\}_{j=1}^n \subset \mathbb{C}^{d\times r}$,

$$S = \sum_{j=1}^{n} T_j T_j^* \qquad \text{is invertible}$$

(1) $\{S^{-1/2}T_j\}_{j=1}^n$ yields

- perfect reconstruction: $\sum_{j=1}^n S^{-1/2}T_j(S^{-1/2}T_j)^* = I$
- not equal norm: $\|S^{-1/2}T_j\|_{HS} \neq \|S^{-1/2}T_k\|_{HS}$ in general...

(2) $\left\{\frac{S^{-1/2}T_j}{\|S^{-1/2}T_j\|_{HS}}\right\}_{j=1}^n$

- equal norm
- no perfect reconstruction: $\sum_{j=1}^n S^{-1/2}T_j(S^{-1/2}T_j)^* \neq I$ in general...

# Outline

# Tyler/Kent's M-estimators

- estimating $\Sigma$ of angular central Gaussian distribution

$$g_\Sigma(x) = \det(\Sigma)^{-1/2}(x^\top \Sigma^{-1} x)^{-d/2}$$

- iterative scheme: $\{x_j\}_{j=1}^n \subset \mathbb{S}^{d-1}$ i.i.d. sample from $g_\Sigma$

$$\Gamma_0 := \frac{1}{d} I, \qquad \Gamma_{k+1} := \frac{\left(\frac{d}{n}\sum_{j=1}^n \frac{x_j x_j^*}{\|\Gamma_k^{1/2} x_j\|^2}\right)^{-1}}{\operatorname{trace}\left(\left(\frac{d}{n}\sum_{j=1}^n \frac{x_j x_j^*}{\|\Gamma_k^{1/2} x_j\|^2}\right)^{-1}\right)}$$

converges towards $\Gamma \approx \Sigma$ (up to scaling).

Kent, Tyler: *Maximum likelihood estimation for the wrapped Cauchy distribution*, J. of Applied Statistics 15 (1988), no. 2, 247-254.

Tyler: *A distribution-free M-estimate of multivariate scatter*, Annals of Statistics 15 (1987), no. 1, 234-251.

Tyler: *Statistical analysis for the angular central Gaussian distribution*, Biometrika 74 (1987), no. 3, 579-590.

## Theorem (E. '13)

*For nice* $\{T_j\}_{j=1}^n \subset \mathbb{C}^{d\times r}$, *the recursive scheme*

$$\Gamma_0 := \frac{1}{d} I, \qquad \Gamma_{k+1} := \frac{\left(\frac{d}{n}\sum_{j=1}^n \frac{T_j T_j^*}{\operatorname{trace}(T_j^* \Gamma_k T_j)}\right)^{-1}}{\operatorname{trace}\left(\left(\frac{d}{n}\sum_{j=1}^n \frac{T_j T_j^*}{\operatorname{trace}(T_j^* \Gamma_k T_j)}\right)^{-1}\right)}$$

*converges towards* $\Gamma \succeq 0$ *and*

$$\left\{ \frac{\Gamma^{1/2} T_j}{\|\Gamma^{1/2} T_j\|_{HS}} \right\}_{j=1}^n$$

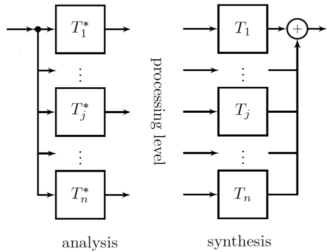
*yields equal norm and perfect reconstruction.*

**E.**: *Preconditioning filter bank decompositions using structured normalized tight frames*, arXiv.

# Isometric filter banks

## Corollary (E. '13)

*For nice* $\{T_j\}_{j=1}^n \subset \mathbb{C}^{d\times r}$,

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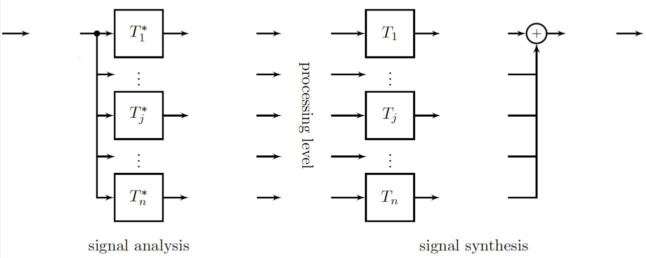


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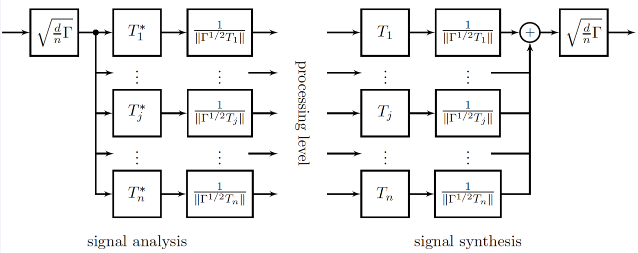


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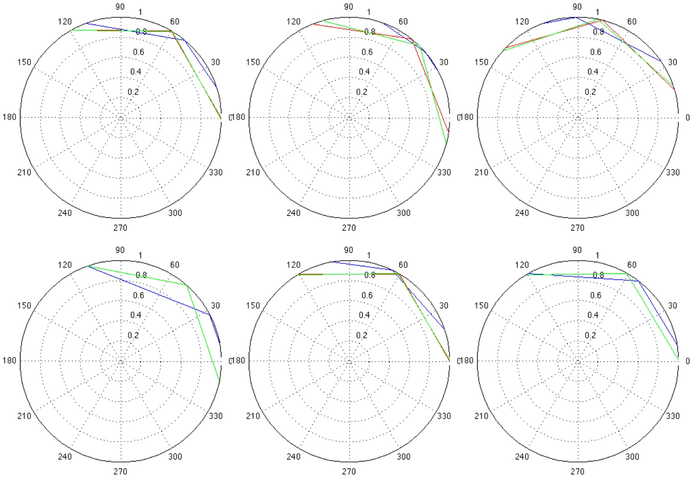
# Example

$d = 3$, $r = 1$

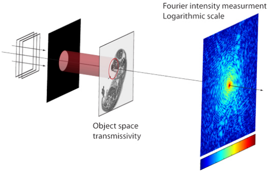


# Outline

# Outlook: diffraction imaging

Let $x \in \mathbb{C}^d$ be unknown

- measure $b_1 = \|P_1 x\|^2, \quad \ldots \quad b_n = \|P_n x\|^2$




$P_j$ orth. rank $k$ projector

AIM: identify $xx^*$

- zonal polynomials on Grassmann manifold $\mathcal{G}$ or, more generally,

$$\mathcal{G} := \{O \operatorname{diag}(\lambda_1, \ldots, \lambda_d) O^* : O \in \mathcal{O}(d)\}$$

- compute moments of "uniform" distribution on $\mathcal{G}$
- compute eigenfunctions of Laplacian on manifolds from the data
- spherical $t$-designs, Grassmannian designs, . . . on $\mathcal{G}$, manifolds. . .

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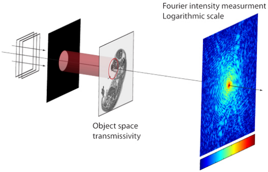


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# Summary

- statistical modeling: granular rod experiments $(\checkmark)$
- covariance estimation: filter bank design $\checkmark$
- moments on manifolds: phase retrieval $\checkmark$
- shape analysis: retinal images