A tractable and interpretable four-parameter family of unimodal distributions on the circle

SHOGO KATO[1] AND M.C. JONES[2]

[1] Institute of Statistical Mathematics, Japan
[2] The Open University, UK

May 21, 2014

Outline

Introduction

TWO-PARAMETER DISTRIBUTIONS

- von Mises distribution,
- cardioid distribution,
- wrapped Cauchy distribution,
- wrapped normal distribution.

These are unimodal models with two parameters, one controlling location and the other concentration.

However these distributions do not allow for variations in skewness and kurtosis.

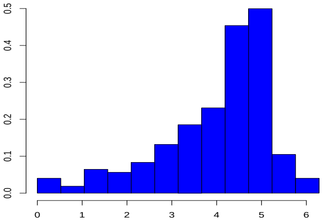


Fig. 1. Histogram of $n = 711$ wind directions at Neuglobsow, Germany, measured hourly between July 1 and 31, 2007.

UNIMODAL FOUR-PARAMETER MODELS

- wrapped stable (Pewsey, 2008)
- direct Batschelet (Abe *et al.*, 2013)
- inverse Batschelet (Jones & Pewsey, 2012)

OUR GOAL

Our goal is to present a unimodal family on the circle which:

(i) has four parameters controlling location, concentration, skewness and kurtosis,

(ii) has wide ranges of skewness and kurtosis,

(iii) includes a well-known two-parameter family as a special case,

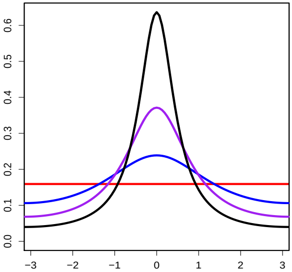
(iv) is mathematically tractable.

Preliminaries

Wrapped Cauchy distribution

Wrapped Cauchy distribution, WC(μ, ρ), is given by the density

$$f(\theta) = \frac{1}{2\pi} \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(\theta-\mu)}, \quad \theta \in [-\pi, \pi); \ \mu \in [-\pi, \pi), \ \rho \in [0, 1).$$



Basic properties

- unimodal and symmetric,
- μ: location parameter,
- ρ: concentration parameter.

Fig. 2. Density of wrapped Cauchy with $\mu = 0$ and: $\rho = 0$, 0.2, 0.4, and 0.6.

Trigonometric Moments

Let Θ be a continuous r.v. on the circle with density f.
Then the kth trigonometric moment (t.m.) of Θ is defined by

$$\phi_\Theta(k) = E(e^{ik\Theta}) = \int_{-\pi}^{\pi} e^{ik\theta} f(\theta)\, d\theta, \quad k = 1, 2, \ldots$$

The cases $k \leq 0$ can be obtained from $\phi_\Theta(0) = 1$, $\phi_\Theta(k) = \overline{\phi_\Theta(-k)}$.

T.M. of the Wrapped Cauchy

Let $\Theta_c \sim \mathrm{WC}(\mu, \rho)$. Then

$$\phi_{\Theta_c}(k) = \left(\rho e^{i\mu}\right)^k, \quad k = 1, 2, \ldots$$

It is known that t.m.'s characterise probability distributions.

An Extension of the Wrapped Cauchy

Wrapped Cauchy: $\phi_{\Theta_c}(k) = \left(\rho e^{i\mu}\right)^k, \quad k = 1, 2, \ldots$

AN EXTENSION

In this talk we propose an extension of the wrapped Cauchy via a characterisation of its t.m.'s.

To achieve this, we first consider the t.m.'s

$$\tilde{\psi}_{\Theta}(k) = \beta e^{i\alpha} \left(\rho e^{i\eta}\right)^k, \quad k = 1, 2, \ldots,$$

where the $\mathbb{R}$-valued parameters, $\alpha, \beta, \eta, \rho$, satisfy certain conditions.

The Main Proposal

DEFINITION (Kato & Jones, revised)

We define a new family by the reparametrised version of $\tilde{\psi}_\Theta$:

$$\psi_\Theta(k) = \gamma\,(\rho e^{i\lambda})^{-1}\left\{\rho e^{i(\mu+\lambda)}\right\}^k, \quad k = 1, 2, \ldots,$$

where the $\mathbb{R}$-valued parameters satisfy certain conditions.

Clearly, ψ_Θ reduces to the t.m. of the wrapped Cauchy if $\lambda = 0$ and $\gamma = \rho$.

Conditions on the Parameters

Our proposal: $\psi_\Theta(k) = \gamma \left(\rho e^{i\lambda}\right)^{-1} \left\{\rho e^{i(\mu+\lambda)}\right\}^k, \quad k = 1, 2, \ldots$

Note that there does not always exist an absolutely continuous distribution whose t.m.'s are equal to ψ_Θ.

THEOREM 1

There exists an absolutely continuous distribution on the circle whose t.m.'s are ψ_Θ iff the parameters satisfy

$$-\pi \leq \mu, \lambda < \pi, \quad 0 \leq \gamma, \rho < 1,$$

$$(\rho \cos \lambda - \gamma)^2 + (\rho \sin \lambda)^2 \leq (1 - \gamma)^2.$$

Probability Density Function

Theorem 2

Let Θ have the absolutely continuous distribution whose t.m.'s are given by ψ_Θ.

Then the probability density function of Θ is a.e. equal to

$$g(\theta) = \frac{1}{2\pi}\left\{1 + 2\gamma\,\frac{\cos(\theta-\mu) - \rho\cos\lambda}{1+\rho^2 - 2\rho\cos(\theta-\mu-\lambda)}\right\}, \quad -\pi \leq \theta < \pi, \quad (1)$$

where the parameters satisfy the conditions given in Theorem 1.

Write $\Theta \sim \mathrm{G}(\mu, \gamma, \rho, \lambda)$ if a r.v. Θ has density (1).

SUMMARY MEASURES

Suppose $\Theta \sim \mathrm{G}(\mu, \gamma, \rho, \lambda)$. Then

(i) mean direction μ_1:

$$\mu_1 \equiv \arg\{E(e^{i\Theta})\} = \mu,$$

(ii) mean resultant length γ_1:

$$\gamma_1 \equiv |E(e^{i\Theta})| = \gamma,$$

(iii) circular kurtosis α_2:

$$\alpha_2 \equiv E[\cos\{2(\Theta - \mu_1)\}] = \gamma\rho\cos\lambda,$$

(iv) circular skewness β_2:

$$\beta_2 \equiv E[\sin\{2(\Theta - \mu_1)\}] = \gamma\rho\sin\lambda.$$

Reparametrisation

REPARAMETRISATION

Given the kurtosis and skewness of our model (1), it is advantageous to reparametrise (ρ, λ) into (α_2, β_2) via

$$\alpha_2 = \gamma\rho\cos\lambda \quad \text{and} \quad \beta_2 = \gamma\rho\sin\lambda.$$

Then the density of our model can be written in terms of four parameters μ, γ, α_2 and β_2 as

$$g(\theta) = \frac{1}{2\pi}\left[1 + \frac{2\gamma^2\{\gamma\cos(\theta-\mu) - \alpha_2\}}{\gamma^2 + \alpha_2^2 + \beta_2^2 - 2\gamma\{\alpha_2\cos(\theta-\mu) + \beta_2\sin(\theta-\mu)\}}\right], \quad -\pi \le \theta < \pi. \quad (2)$$

The parameter μ controls mean direction, γ mean resultant length, α_2 circular kurtosis, and β_2 circular skewness.

The conditions on the parameters of the reparametrised model (2) are:

$$-\pi \leq \mu < \pi, \quad 0 \leq \gamma < 1, \quad (\alpha_2, \beta_2) \neq (\gamma, 0) \quad \text{and}$$

$$(\alpha_2 - \gamma^2)^2 + \beta_2^2 \leq \gamma^2(1-\gamma)^2.$$

LEMMA 1

For fixed $\gamma \, (= \gamma_1)$:

(i) $\sup_{\rho,\lambda} \alpha_2(\rho, \lambda) = \gamma,$
$\min_{\rho,\lambda} \alpha_2(\rho, \lambda) = \gamma(2\gamma - 1),$

(ii) $\max_{\rho,\lambda} \beta_2(\rho, \lambda) = \gamma(1-\gamma),$
$\min_{\rho,\lambda} \beta_2(\rho, \lambda) = -\gamma(1-\gamma).$

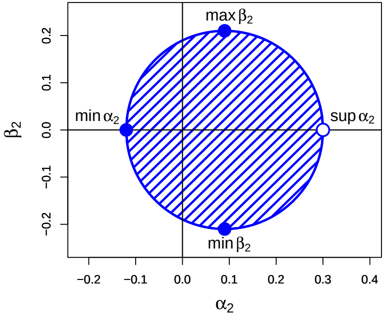


Fig. 3. Extent of circular kurtosis α_2 and circular skewness β_2 for $\gamma = 0.3$.

Shapes of Density (2)

$$g(\theta) = \frac{1}{2\pi}\left[1 + \frac{2\gamma^2\{\gamma\cos(\theta-\mu)-\alpha_2\}}{\gamma^2+\alpha_2^2+\beta_2^2-2\gamma\{\alpha_2\cos(\theta-\mu)+\beta_2\sin(\theta-\mu)\}}\right]. \quad (2)$$

THEOREM 3

(i) Density (2) is unimodal if $\gamma > 0$ and uniform if $\gamma = 0$.

(ii) Density (2) is symmetric $\iff$ $\beta_2 = 0$.

(iii) The mode and antimode of density (2) with $\gamma > 0$ can be expressed in closed form.

Plot of Density (2)

$$g(\theta) = \frac{1}{2\pi}\left[1 + \frac{2\gamma^2\{\gamma\cos(\theta-\mu)-\alpha_2\}}{\gamma^2+\alpha_2^2+\beta_2^2-2\gamma\{\alpha_2\cos(\theta-\mu)+\beta_2\sin(\theta-\mu)\}}\right]. \quad (2)$$

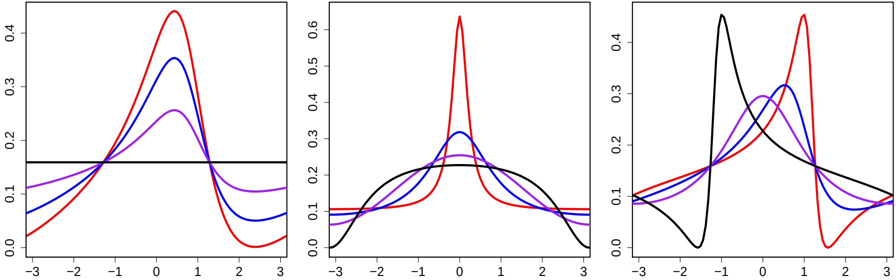


Fig. 4. Density (2) for $\mu = 0$ and:

(L) $\alpha_2 = 0.4\gamma\cos(\pi/4)$, $\beta_2 = 0.4\gamma\sin(\pi/4)$, and $\gamma = 0, 0.2, 0.4, 0.58$;

(C) $\gamma = 0.3$, $\beta_2 = 0$, and $\alpha_2 = -0.12, 0, 0.12, 0.24$;

(R) $\gamma = 0.3$, $\alpha_2 = 0.09$, and $\beta_2 = -0.21, 0, 0.165, 0.21$.

Submodels

Case 1: wrapped Cauchy distribution ($\alpha_2 = \gamma^2,\ \beta_2 = 0$)

Case 2: cardioid distribution ($\alpha_2 = \beta_2 = 0$)

$$g(\theta) = \frac{1}{2\pi}\left\{1 + 2\gamma\cos(\theta - \mu)\right\}.$$

Case 3: The sine-skewed Cauchy distribution

(Umbach & Jammalamadaka, 2009; Abe & Pewsey, 2011)

($\rho = \gamma\cos\lambda$ in the original parameterisation)

$$g(\theta) = \left\{1 + \check{\lambda}\sin(\theta - \mu)\right\}\frac{1}{2\pi}\frac{1 - \rho^2}{1 + \rho^2 - 2\rho\cos(\theta - \mu)},$$

where $\check{\lambda} = 2\,\mathrm{sign}(\lambda)\{\gamma^2 - \rho^2\}^{1/2}/(1 - \rho^2)$.

Case 4: The three-parameter symmetric submodel ($\beta_2 = 0$)

Case 5 (new): A three-parameter asymmetric submodel ($\alpha_2 = \gamma^2$)

In addition, some other models such as a point distribution ($\gamma \to 1$) and circular uniform ($\gamma = 0$) are included as special cases.

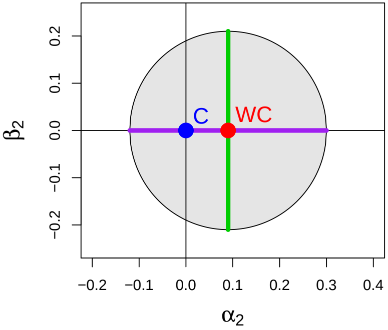


Fig. 5.

Domain of (α_2, β_2) (gray) with $\gamma = 0.3$.

Positions of submodels are:

WC: wrapped Cauchy,

C: cardioid,

—: 3-parameter symmetric,

—: 3-parameter asymmetric.

Some Other Properties

The following can be derived using the original parameterisation.

Theorem 4

(i) $\Theta_1 \sim \mathrm{G}(\mu_1, \gamma_1, \rho_1, \lambda_1),\ \Theta_2 \sim \mathrm{G}(\mu_2, \gamma_2, \rho_2, \lambda_2),\ \Theta_1 \perp \Theta_2$
$$\Longrightarrow \quad \Theta_1 + \Theta_2 \sim \mathrm{G}(\mu_1 + \mu_2, \gamma_1\gamma_2, \rho_1\rho_2, \lambda_1 + \lambda_2).$$

(ii) $\Theta \sim \mathrm{G}(\mu, \gamma, \rho, \lambda)$
$$\Longrightarrow \ n\Theta \pmod{2\pi} \sim \mathrm{G}(n\mu + (n-1)\lambda, \gamma\rho^{n-1}, \rho^n, \lambda), \quad n \in \mathbb{N}.$$

(iii) $\Theta \sim \mathrm{G}(\mu, \gamma, \rho, \lambda) \ \Longrightarrow -\Theta \sim \mathrm{G}(-\mu, \gamma, \rho, -\lambda).$

Theorem 5

$\gamma \leq \rho,\ \lambda = 0 \ \Longrightarrow$ Distribution (1) is infinitely divisible.

Parameter Estimation

Let $\Theta_1, \ldots, \Theta_n \sim i.i.d.$ $\mathrm{G}(\mu, \gamma, \alpha_2, \beta_2)$.

METHOD OF MOMENTS ESTIMATION

Method of moments estimators based on the t.m.'s are

$$\hat{\mu} = \arg(S_{1n}), \quad \hat{\gamma} = |S_{1n}|,$$

$$(\hat{\alpha}_2, \hat{\beta}_2) = \begin{cases} (a_2, b_2), & (a_2, b_2) \in D_{\hat{\gamma}}, \\ \left(\frac{\hat{\gamma}(1-\hat{\gamma})(a_2-\hat{\gamma}^2)}{\sqrt{(a_2-\hat{\gamma}^2)^2+b_2^2}} + \hat{\gamma}^2, \frac{\hat{\gamma}(1-\hat{\gamma})b_2}{\sqrt{(a_2-\hat{\gamma}^2)^2+b_2^2}} \right), & (a_2, b_2) \notin D_{\hat{\gamma}}, \end{cases}$$

where $S_{1n} = n^{-1} \sum_{j=1}^{n} e^{i\Theta_j}$,

$$a_2 = n^{-1} \textstyle\sum_j \cos\{2(\Theta_j - \hat{\mu})\}, \; b_2 = n^{-1} \sum_j \sin\{2(\Theta_j - \hat{\mu})\},$$

$$D_\gamma = \{(x, y) \in \mathbb{R}^2 \, ; \, (x - \gamma^2)^2 + y^2 \leq \gamma^2(1 - \gamma)^2\}.$$

Let $\Theta_1, \ldots, \Theta_n \sim i.i.d.$ $\mathrm{G}(\mu, \gamma, \alpha_2, \beta_2)$.

MAXIMUM LIKELIHOOD ESTIMATION

The log-likelihood function for $(\theta_1, \ldots, \theta_n)$ is

$$\ell = C + \sum_{j=1}^{n} \log\left[1 + \frac{2\gamma^2\{\gamma\cos(\theta_j - \mu) - \alpha_2\}}{\gamma^2 + \alpha_2^2 + \beta_2^2 - 2\gamma\{\alpha_2\cos(\theta_j - \mu) + \beta_2\sin(\theta_j - \mu)\}}\right].$$

- The maximum likelihood estimates of $(\mu, \gamma, \alpha_2, \beta_2)$ should be obtained numerically.
- Our simulation study suggests:
  - (i) method of moments estimates provide useful starting values,
  - (ii) maximum likelihood estimation is very fast.

Comparison with Inverse Batschelet (IB) Distribution

A special case of the IB family (Jones & Pewsey, 2012):

$$g_\nu(\theta) \propto f\{t_\nu(\theta)\}, \quad -\pi \le \theta < \pi,$$

where f: density of Jones & Pewsey's (2005) 3-parameter symmetric family,
$t_\nu(\theta) = t_{1,\nu}^{-1}(\theta), \quad t_{1,\nu}(\theta) = \theta - \nu - \nu\cos\theta.$

Common properties of model (1) and IB

- unimodal family having 4 parameters with clear interpretation,
- tractable density,
- inclusion of WC and cardioid.

Model (1) only

- simple t.m.'s,
- fast parameter estimation.

IB only

- inclusion of von Mises,
- parameter orthogonality.

Conclusions

PROPERTIES OF THE PROPOSED MODEL

- unimodal density expressed in closed form,
- simple t.m.'s and summary measures,
- four parameters controlling location, concentration, kurtosis and skewness,
- wide ranges of kurtosis and skewness,
- inclusion of wrapped Cauchy, cardioid, sine-skewed Cauchy, 3-parameter symmetric and asymmetric submodels, etc.,
- closure properties and infinite divisibility,
- fast parameter estimation.

References I

[1] ABE, T. & PEWSEY, A. (2011). Sine-skewed circular distributions. *Statistical Papers*, **52**, 683–707.

[2] ABE, T., PEWSEY, A. & SHIMIZU, K. (2013). Extending circular distributions through transformation of argument. *Annals of the Institute of Statistical Mathematics*, **65**, 833–858.

[3] JONES, M.C. & PEWSEY, A. (2005). A family of symmetric distributions on the circle. *Journal of the American Statistical Association*, **100**, 1422–1428.

[4] JONES, M.C. & PEWSEY, A. (2012). Inverse Batschelet distributions for circular data. *Biometrics*, **68**, 183–193.

References II

[5] KATO, S. & JONES, M.C. A tractable and interpretable four-parameter family of unimodal distributions on the circle. Revised.

[6] PEWSEY, A. (2008). The wrapped stable family of distributions as a flexible model for circular data. *Computational Statistics and Data Analysis*, **52**, 1516–1523.

[7] UMBACH, D. & JAMMALAMADAKA, S.R. (2009). Building asymmetry into circular distributions. *Statistics and Probability Letters*, **79**, 659–663.